100 years of Zermelo's axiom of choice: what was the problem with it?

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Cantor conceived set theory in a sequence of six papers published in the Mathematische Annalen during the five year period 1879-1884. In the fifth of these papers, published in 1883,[1] he stated as a law of thought (Denkgesetz) that every set can be well-ordered or, more precisely, that it is always possible to bring any well-defined set into the form of a well-ordered set. Now to call it a law of thought was implicitly to claim self-evidence for it, but he must have given up that claim at some point, because in the 1890's he made an unsuccessful attempt at demonstrating the well-ordering principle.[2]

The first to succeed in doing so was Zermelo,[3] although, as a prerequisite of the demonstration, he had to introduce a new principle, which came to be called the principle of choice (Prinzip der Auswahl) respectively the axiom of choice (Axiom der Auswahl) in his two papers from 1908.[4,5] His first paper on the subject, published in 1904, consists of merely three pages, excerpted by Hilbert from a letter which he had received from Zermelo. The letter is dated 24 September 1904, and the excerpt begins by saying that the demonstration came out of discussions with Erhard Schmidt during the preceding week, which means that we can safely date the appearance of the axiom of choice and the demonstration of the well-ordering theorem to September 1904.

Brief as it was, Zermelo's paper gave rise to what is presumably the most lively discussion among mathematicians on the validity, or acceptability, of a mathematical axiom that has ever taken place. Within a couple of

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[1] G. Cantor, Über unendliche lineare Punktmannigfaltigkeiten. Nr. 5, *Math. Annalen*, Vol. 21, 1883, pp. 545-591. Reprinted in *Gesammelte Abhandlungen*, Edited by E. Zermelo, Springer-Verlag, Berlin, 1932, pp. 165-208.

[2] G. H. Moore, *Zermelo's Axiom of Choice: Its Origins, Development, and Influence*, Springer-Verlag, New York, 1982, p. 51.

[3] E. Zermelo, Beweis, daß jede Menge wohlgeordnet werden kann. (Aus einem an Herrn Hilbert gerichteten Briefe.), *Math. Annalen*, Vol. 59, pp. 514-516.

[4] E. Zermelo, Neuer Beweis für die Möglichkeit einer Wohlordnung, *Math. Annalen*, Vol. 65, 1908, pp. 107-128.

[5] E. Zermelo, Untersuchungen über die Grundlagen der Mengenlehre. I, *Math. Annalen*, Vol. 65, 1908, pp. 261-281.

years, written contributions to this discussion had been published by Felix Bernstein, Schoenflies, Hamel, Hessenberg and Hausdorff in Germany, Baire, Borel, Hadamard, Lebesgue, Richard and Poincaré in France, Hobson, Hardy, Jourdain and Russell in England, Julius König in Hungary, Peano in Italy and Brouwer in the Netherlands.[6] Zermelo responded to those of these contributions that were critical, which was a majority, in a second paper from 1908. This second paper also contains a new proof of the well-ordering theorem, less intuitive or less perspicuous, it has to be admitted, than the original proof, as well as a new formulation of the axiom of choice, a formulation which will play a crucial role in the following considerations.

Despite the strength of the initial opposition against it, Zermelo's axiom of choice gradually came to be accepted mainly because it was needed at an early stage in the development of several branches of mathematics, not only set theory, but also topology, algebra and functional analysis, for example. Towards the end of the thirties, it had become firmly established and was made part of the standard mathematical curriculum in the form of Zorn's lemma.[7]

The intuitionists, on the other hand, rejected the axiom of choice from the very beginning, Baire, Borel and Lebesgue were all critical of it in their contributions to the correspondence which was published under the title Cinq lettres sur la théorie des ensembles in 1905.[8] Brouwer's thesis from 1907 contains a section on the well-ordering principle in which is treated in a dismissive fashion ("of course there is no motivation for this at all") and in which, following Borel,[9] he belittles Zermelo's proof of it from the axiom of choice.[10] No further discussion of the axiom of choice seems to be found in either Brouwer's or Heyting's writings. Presumably, it was regarded by them as a prime example of a nonconstructive principle.

It therefore came as a surprise when, as late as in 1967, Bishop stated,

> A choice function exists in constructive mathematics, because a choice is *implied by the very meaning of existence*,[11]

[6]G. H. Moore, op. cit., pp. 92-137.

[7]M. Zorn, A remark on method in transfinite algebra, *Bull. Amer. Math. Soc.*, Vol. 41, 1935, pp. 667-670.

[8]R. Baire, É. Borel, J. Hadamard and H. Lebesgue, Cinq lettres sur la théorie des ensembles, *Bull. Soc. Math. France*, Vol. 33, 1905, pp. 261-273.

[9]É. Borel, Quelques remarques sur les principes de la théorie des ensembles, *Math. Annalen*, Vol. 60, 1905, pp. 194-195.

[10]L. E. J. Brouwer, *Over de grondslagen der wiskunde*, Maas & van Suchtelen, Amsterdam, 1907. English translation in *Collected Works*, Vol. 1, Edited by A. Heyting, North-Holland, Amsterdam, 1975, pp. 11-101.

[11]E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967, p. 9.

although, in the terminology that he himself introduced in the subsequent chapter, he ought to have said "choice operation" rather than "choice function". What he had in mind was clearly that the truth of

$$(\forall i : I)(\exists x : S)A(i, x) \to (\exists f : I \to S)(\forall i : I)A(i, f(i))$$

and even, more generally,

$$(\forall i : I)(\exists x : S_i)A(i, x) \to (\exists f : \prod_{i:I} S_i)(\forall i : I)A(i, f(i))$$

becomes evident almost immediately upon remembering the Brouwer-Heyting-Kolmogorov interpretation of the logical constants, which means that it might as well have been observed already in the early thirties. And it is this intuitive justification that was turned into a formal proof in constructive type theory, a proof that effectively uses the strong rule of ∃-elimination that it became possible to formulate as a result of having made the proof objects appear in the system itself and not only in its interpretation.

In 1975, soon after Bishop's vindication of the constructive axiom of choice, Diaconescu proved that, in topos theory, the law of excluded middle follows from the axiom of choice.[12] Now, topos theory being an intuitionistic theory, albeit impredicative, this is on the surface of it incompatible with Bishop's observation because of the constructive inacceptability of the law of excluded middle. There is therefore a need to investigate how the constructive axiom of choice, validated by the Brouwer-Heyting-Kolmogorov interpretation, is related to Zermelo's axiom of choice on the one hand and to the topos-theoretic axiom of choice on the other.

To this end, using constructive type theory as our instrument of analysis, let us simply try to prove Zermelo's axiom of choice. This attempt should of course fail, but in the process of making it we might at least be able to discover what it is that is really objectionable about it. So what was Zermelo's axiom of choice? In the original paper from 1904, he gave to it the following formulation,

> *Jeder Teilmenge M' denke man sich ein beliebiges Element m'_1 zugeordnet, das in M' selbst vorkommt und das „ausgezeichnete" Element von M' genannt werden möge.*[13]

Here M' is an arbitrary subset, which contains at least one element, of a given set M. What is surprising about this formulation is that there is

[12]R. Diaconescu, Axiom of choice and complementation, *Proc. Amer. Math. Soc.*, Vol. 51, 1975, pp. 176-178.

[13]E. Zermelo, op. cit., footnote 3, p. 514.

nothing objectionable about it from a constructive point of view. Indeed, the distinguished element m'_1 can be taken to be the left projection of the proof of the existential proposition $(\exists x : M)M'(x)$, which says that the subset M' of M contains at least one element. This means that one would have to go into the demonstration of the well-ordering theorem in order to determine exactly what are its nonconstructive ingredients.

Instead of doing that, I shall turn to the formulation of the axiom of choice that Zermelo favoured in his second paper on the well-ordering theorem from 1908,

> Axiom. *Eine Menge S, welche in eine Menge getrennter Teile A, B, C, ... zerfällt, deren jeder mindestens ein Element enthält, besitzt mindestens eine Untermenge S_1, welche mit jedem der betrachteten Teile A, B, C, ... genau ein Element gemein hat.*[14]

Formulated in this way, Zermelo's axiom of choice turns out to coincide with the multiplicative axiom, which Whitehead and Russell had found indispensable for the development of the theory of cardinals.[15,16] The type-theoretic rendering of this formulation of the axiom of choice is straightforward, once one remembers that a basic set in the sense of Cantorian set theory corresponds to an extensional set, that is, a set equipped with an equivalence relation, in type theory, and that a subset of an extensional set is interpreted as a propositional function which is extensional with respect to the equivalence relation in question. Thus the data of Zermelo's 1908 formulation of the axiom of choice are a set S, which comes equipped with an equivalence relation $=_S$, and a family $(A_i)_{i:I}$ of propositional functions on S satisfying the following properties,

(1) $x =_S y \rightarrow (A_i(x) \leftrightarrow A_i(y))$ (extensionality),

(2) $i =_I j \rightarrow (\forall x : S)(A_i(x) \leftrightarrow A_j(x))$ (extensionality of the dependence on the index),

(3) $(\exists x : S)(A_i(x) \,\&\, A_j(x)) \rightarrow i =_I j$ (mutual exclusiveness),

(4) $(\forall x : S)(\exists i : I)A_i(x)$ (exhaustiveness),

(5) $(\forall i : I)(\exists x : S)A_i(x)$ (nonemptiness).

[14] E. Zermelo, op. cit., footnote 4, p. 110.

[15] A. N. Whitehead, On cardinal numbers, *Amer. J. Math.*, Vol. 24, 1902, pp. 367-394.

[16] B. Russell, On some difficulties in the theory of transfinite numbers and order types, *Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 29-53.

Given these data, the axiom guarantees the existence of a propositional function S_1 on S such that

(6) $x =_S y \to (S_1(x) \leftrightarrow S_1(y))$ (extensionality),

(7) $(\forall i : I)(\exists ! x : S)(A_i \cap S_1)(x)$ (uniqueness of choice).

The obvious way of trying to prove (6) and (7) from (1)-(5) is to apply the type-theoretic (constructive, intensional) axiom of choice to (5), so as to get a function $f : I \to S$ such that

$$(\forall i : I)A_i(f(i)),$$

and then define S_1 by the equation

$$S_1 = \{f(j) \mid j : I\} = \{x \mid (\exists j : I)(f(j) =_S x)\}.$$

Defined in this way, S_1 is clearly extensional, which is to say that it satisfies (6). What about (7)? Since the proposition

$$(A_i \cap S_1)(f(i)) = A_i(f(i)) \ \& \ S_1(f(i))$$

is clearly true, so is

$$(\forall i : I)(\exists x : S)(A_i \cap S_1)(x),$$

which means that only the uniqueness condition remains to be proved. To this end, assume that the proposition

$$(A_i \cap S_1)(x) = A_i(x) \ \& \ S_1(x)$$

is true, that is, that the two propositions

$$\begin{cases} A_i(x), \\ S_1(x) = (\exists j : I)(f(j) =_S x), \end{cases}$$

are both true. Let $j : I$ satisfy $f(j) =_S x$. Then, since $(\forall i : I)A_i(f(i))$ is true, so is $A_j(f(j))$. Hence, by the extensionality of A_j with respect to $=_S$, $A_j(x)$ is true, which, together with the assumed truth of $A_i(x)$, yields $i =_I j$ by the mutual exclusiveness of the family of subsets $(A_i)_{i:I}$. At this stage, in order to conclude that $f(i) =_S x$, we need to know that the choice function f is extensional, that is, that

$$i =_I j \to f(i) =_S f(j).$$

This, however, is not guaranteed by the constructive, or intensional, axiom of choice which follows from the strong rule of $\exists$-elimination in type theory.

Thus our attempt to prove Zermelo's axiom of choice has failed, as was to be expected.

On the other hand, we have succeeded in proving that Zermelo's axiom of choice follows from the extensional axiom of choice

$$(\forall i : I)(\exists x : S)A_i(x) \to (\exists f : I \to S)(\mathrm{Ext}(f) \;\&\; (\forall i : I)A_i(f(i))),$$

which I shall call ExtAC, where

$$\mathrm{Ext}(f) = (\forall i, j : I)(i =_I j \to f(i) =_S f(j)).$$

The only trouble with it is that it lacks the evidence of the intensional axiom of choice, which does not prevent one from investigating its consequences, of course.

THEOREM I. *The following are equivalent in constructive type theory:*
(i) The extensional axiom of choice.
(ii) Zermelo's axiom of choice.
(iii) Epimorphisms split, that is, every surjective extensional function has an extensional right inverse.
(iv) Unique representatives can be picked from the equivalence classes of any given equivalence relation.

Of these four equivalent statements, (iii) is the topos-theoretic axiom of choice, which is thus equivalent, not to the constructively valid type-theoretic axiom of choice, but to Zermelo's axiom of choice.

Proof. We shall prove the implications (i)→(ii)→(iii)→(iv)→(i) in this order.

(i)→(ii). This is precisely the result of the considerations prior to the formulation of the theorem.

(ii)→(iii). Let $S, =_S$ and $I, =_I$ be two extensional sets, and let $f : S \to I$ be an extensional and surjective mapping between them. By definition, put

$$A_i = f^{-1}(i) = \{x | f(x) =_I i\}.$$

Then

(1) $x =_S y \to (A_i(x) \leftrightarrow A_i(y))$

by the assumed extensionality of f,

(2) $i =_I j \to (\forall x : S)(A_i(x) \leftrightarrow A_j(x))$

since $f(x) =_I i$ is equivalent to $f(x) =_I j$ provided that $i =_I j$,

(3) $(\exists x : S)(A_i(x) \;\&\; A_j(x)) \to i =_I j$

since $f(x) =_I i$ and $f(x) =_I j$ together imply $i =_I j$,

(4) $(\forall x : S)(\exists i : I)A_i(x)$

since $A_{f(x)}(x)$ for any $x : S$, and

(5) $(\forall i : I)(\exists x : S)A_i(x)$

by the assumed surjectivity of the function f. Therefore we can apply Zermelo's axiom of choice to get a subset S_1 of S such that

$$(\forall i : I)(\exists! x : S)(A_i \cap S_1)(x).$$

The constructive, or intensional, axiom of choice, to which we have access in type theory, then yields $g : I \to S$ such that $(A_i \cap S_1)(g(i))$, that is,

$$(f(g(i)) =_I i) \;\&\; S_1(g(i)),$$

so that g is a right inverse of f, and

$$(A_i \cap S_1)(x) \to g(i) =_S x.$$

It remains only to show that g is extensional. So assume $i, j : I$. Then we have

$$(A_i \cap S_1)(g(i))$$

as well as

$$(A_j \cap S_1)(g(j))$$

so that, if also $i =_I j$,

$$(A_i \cap S_1)(g(j))$$

by the extensional dependence of A_i on the index i. The uniqueness property of $A_i \cap S_1$ permits us to now conclude $g(i) =_S g(j)$ as desired.

(iii)$\to$(iv). Let I be a set equipped with an equivalence relation $=_I$. Then the identity function on I is an extensional surjection from I, Id_I to $I, =_I$, since any function is extensional with respect to the identity relation. Assuming that epimorphisms split, we can conclude that there exists a function $g : I \to I$ such that

$$g(i) =_I i$$

and

$$i =_I j \to \mathrm{Id}_I(g(i), g(j)),$$

which is to say that g has the miraculous property of picking a unique representative from each equivalence class of the given equivalence relation $=_I$.

(iv)$\to$(i). Let $I, =_I$ and $S, =_S$ be two sets, each equipped with an equivalence relation, and let $(A_i)_{i:I}$ be a family of extensional subsets of S,

$$x =_S y \to (A_i(x) \leftrightarrow A_i(y)),$$

which depends extensionally on the index i,

$$i =_I j \to (\forall x : S)(A_i(x) \leftrightarrow A_j(x)).$$

Furthermore, assume that

$$(\forall i : I)(\exists x : S)A_i(x)$$

holds. By the intensional axiom of choice, valid in constructive type theory, we can conclude that there exists a choice function $f : I \to S$ such that

$$(\forall i : I)A_i(f(i)).$$

This choice function need not be extensional, of course, unless $=_I$ is the identity relation on the index set I. But, applying the miraculous principle of picking a unique representative of each equivalence class to the equivalence relation $=_I$, we get a function $g : I \to I$ such that

$$g(i) =_I i$$

and

$$i =_I j \to \mathrm{Id}_I(g(i), g(j)).$$

Then $f \circ g : I \to S$ becomes extensional,

$$i =_I j \to \mathrm{Id}_I(g(i), g(j)) \to \underbrace{f(g(i))}_{(f\circ g)(i)} =_S \underbrace{f(g(j))}_{(f\circ g)(j)}.$$

Moreover, from $(\forall i : I)A_i(f(i))$, it follows that

$$(\forall i : I)A_{g(i)}(f(g(i))).$$

But

$$g(i) =_I i \to (\forall x : S)(A_{g(i)}(x) \leftrightarrow A_i(x)),$$

so that

$$(\forall i : I)A_i(\underbrace{f(g(i))}_{(f\circ g)(i)}).$$

Hence $f \circ g$ has become an extensional choice function, which means that the extensional axiom of choice is satisfied.

Another indication that the extensional axiom of choice is the correct type-theoretic rendering of Zermelo's axiom of choice comes from constructive set theory. Peter Aczel has shown how to interpret the language of Zermelo-Fraenkel set theory in constructive type theory, this interpretation being the natural constructive version of the cumulative hierarchy, and investigated what set-theoretical principles that become validated under that interpretation.[17] But one may also ask, conversely, what principle, or principles, that have to be adjoined to constructive type theory in order to validate a specific set-theoretical axiom. In particular, this may be asked about the formalized version of the axiom of choice that Zermelo made part of his own axiomatization of set theory. The answer is as follows.

THEOREM II. *When constructive type theory is strengthened by the extensional axiom of choice, the set-theoretical axiom of choice becomes validated under the Aczel interpretation.*

Proof. The set-theoretical axiom of choice says that, for any two iterative sets α and β and any relation R between iterative sets,

$$(\forall x \in \alpha)(\exists y \in \beta)R(x, y) \to (\exists \phi : \alpha \to \beta)(\forall x \in \alpha)R(x, \phi(x)).$$

The Aczel interpretation of the left-hand member of this implication is

$$(\forall x : \bar{\alpha})(\exists y : \bar{\beta})R(\tilde{\alpha}(x), \tilde{\beta}(x)),$$

which yields

$$(\exists f : \bar{\alpha} \to \bar{\beta})(\forall x : \bar{\alpha})R(\tilde{\alpha}(x), \tilde{\beta}(f(x)))$$

by the type-theoretic axiom of choice. Now, put

$$\phi = \{\langle \tilde{\alpha}(x), \tilde{\beta}(f(x))\rangle | x : \bar{\alpha}\}$$

by definition. We need to prove that ϕ is a function from α to β in the sense of constructive set theory, that is,

$$\tilde{\alpha}(x) = \tilde{\alpha}(x') \to \tilde{\beta}(f(x)) = \tilde{\beta}(f(x')).$$

Define the equivalence relations

$$(x =_{\bar{\alpha}} x') = (\tilde{\alpha}(x) = \tilde{\alpha}(x'))$$

and

$$(y =_{\bar{\beta}} y') = (\tilde{\beta}(y) = \tilde{\beta}(y'))$$

[17] P. Aczel, The type theoretic interpretation of constructive set theory, *Logic Colloquium '77*, Edited by A. Macintyre, L. Pacholski and J. Paris, North-Holland, Amsterdam, 1978, pp. 55-66.

on $\bar{\alpha}$ and $\bar{\beta}$, respectively. By the extensional axiom of choice in type theory, the choice function $f : \bar{\alpha} \to \bar{\beta}$ can be taken to be extensional with respect to these two equivalence relations,

$$x =_{\bar{\alpha}} x' \to f(x) =_{\bar{\beta}} f(x'),$$

which ensures that ϕ, defined as above, is a function from α to β in the sense of constructive set theory.

COROLLARY. *When constructive type theory (including one universe and the W-operation) is strengthened by the extensional axiom of choice, it interprets all of* ZFC.

Proof. We already know from Aczel that ZF is equivalent to CZF + EM.[18] Hence ZFC is equivalent to CZF + EM + AC. But, by Diaconescu's theorem as transferred to constructive set theory by Goodman and Myhill, the law of excluded middle follows from the axiom of choice in the context of constructive set theory.[19] It thus suffices to interpret CZF + AC in CTT + ExtAC, and this is precisely what the Aczel interpretation does, by the previous theorem.

Another way of reaching the same conclusion is to interchange the order of the last two steps in the proof just given, arguing instead that ZFC = CZF + EM + AC is interpretable in CTT + EM + ExtAC by the previous theorem, and then appealing to the type-theoretical version of Diaconescu's theorem, according to which the law of excluded middle follows from the extensional axiom of choice in the context of constructive type theory.[20] The final conclusion is anyhow that ZFC is interpretable in CTT+ExtAC.

When Zermelo's axiom of choice is formulated in the context of constructive type theory instead of Zermelo-Fraenkel set theory, it appears as ExtAC, the extensional axiom of choice

$$(\forall i : I)(\exists x : S)A(i, x) \to (\exists f : I \to S)(\mathrm{Ext}(f) \;\&\; (\forall i : I)A(i, f(i))),$$

where

$$\mathrm{Ext}(f) = (\forall i, j : I)(i =_I j \to f(i) =_S f(j)),$$

and it then becomes manifest what is the problem with it: it breaks the principle that you cannot get something from nothing. Even if the relation $A(i, x)$ is extensional with respect to its two arguments, the truth of the

[18]P. Aczel, op. cit., p. 59.

[19]N. D. Goodman and J. Myhill, Choice implies excluded middle, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, Vol. 24, 1978, p. 461.

[20]S. Lacas and B. Werner, Which choices imply the Excluded Middle? About Diaconescu's trick in Type Theory, Unpublished, 1999, pp. 9-10. I am indebted to Jesper Carlström for providing me with this reference.

antecedent $(\forall i : I)(\exists x : S)A(i,x)$, which does guarantee the existence of a choice function $f : I \to S$ satisfying $(\forall i : I)A(i, f(i))$, is not enough to guarantee the extensionality of the choice function, that is, the truth of $\mathrm{Ext}(f)$. Thus the problem with Zermelo's axiom of choice is not the existence of the choice function but its extensionality, and this is not visible within an extensional framework, like Zermelo-Fraenkel set theory, where all functions are by definition extensional.

If we want to ensure the extensionality of the choice function, the antecedent clause of the extensional axiom of choice has to be strengthened. The natural way of doing this is to replace ExtAC by AC!, the axiom of unique choice, or no choice,

$$(\forall i : I)(\exists! x : S)A(i,x) \to (\exists f : I \to S)(\mathrm{Ext}(f) \;\&\; (\forall i : I)A(i, f(i))),$$

which is as valid as the intensional axiom of choice. Indeed, assume $(\forall i : I)(\exists! x : S)A(i,x)$ to be true. Then, by the intensional axiom of choice, there exists a choice function $f : I \to S$ satisfying $(\forall i : I)A(i, f(i))$. Because of the uniqueness condition, such a function $f : I \to S$ is necessarily extensional. For suppose that $i, j : I$ are such that $i =_I j$ is true. Then $A(i, f(i))$ and $A(j, f(j))$ are both true. Hence, by the extensionality of $A(i,x)$ in its first argument, so is $A(i, f(j))$. The uniqueness condition now guarantees that $f(i) =_S f(j)$, that is, that $f : I \to S$ is extensional. The axiom of unique choice AC! may be considered as the valid form of extensional choice, as opposed to ExtAC, which lacks justification.

The inability to distinguish between the intensional and the extensional axiom of choice has led to one's taking the need for the axiom of choice in proving that the union of a countable sequence of countable sets is again countable, that the real numbers, defined as Cauchy sequences of rational numbers, are Cauchy complete, etc., as a justification of Zermelo's axiom of choice. As Zermelo himself wrote towards the end of the short paper in which he originally introduced the axiom of choice,

> Dieses logische Prinzip läßt sich zwar nicht auf ein noch einfacheres zurückführen, wird aber in der mathematischen Deduktion überall unbedenklich angewendet.[21]

What Zermelo wrote here about the omnipresent, and often subconscious, use of the axiom of choice in mathematical proofs is incontrovertible, but it concerns the constructive, or intensional, version of it, which follows almost immediately from the strong rule of existential elimination. It cannot be

[21]E. Zermelo, op. cit., footnote 3, p. 516.

taken as a justification of his own version of the axiom of choice, including as it does the extensionality of the choice function.

Within an extensional foundational framework, like topos theory or constructive set theory, it is not wholly impossible to formulate a counterpart of the constructive axiom of choice, despite of its intensional character, but it becomes complicated. In topos theory, there is the axiom that there are enough projectives, which is to say that every object is the surjective image of a projective object, and, in constructive set theory, Aczel has introduced the analogous axiom that every set has a base.[22] Roughly speaking, this is to say that every set is the surjective image of a set for which the axiom of choice holds. The technical complication of these axioms speaks to my mind for an intensional foundational framework, like constructive type theory, in which the intuitive argument why the axiom of choice holds on the Brouwer-Heyting-Kolmogorov interpretation is readily formalized, and in which whatever extensional notions that are needed can be built up, in agreement with the title of Martin Hofmann's thesis, Extensional Constructs in Intensional Type Theory.[23] Extensionality does not come for free.

Finally, since this is only a couple of weeks from the centenary of Zermelo's first formulation of the axiom of choice, it may not be out of place to remember the crucial role it has played for the formalization of both Zermelo-Fraenkel set theory and constructive type theory. In the case of set theory, there was the need for Zermelo of putting his proof of the well-ordering theorem on a formally rigorous basis, whereas, in the case of type theory, there was the intuitively convincing argument which made axiom of choice evident on the constructive interpretation of the logical operations, an argument which nevertheless could not be faithfully formalized in any then existing formal system.

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[22] P. Aczel, The type theoretic interpretation of constructive set theory: choice principles, *The L. E. J. Brouwer Centenary Symposium*, Edited by A. S. Troelstra and D. van Dalen, North-Holland, Amsterdam, 1982, pp. 1-40.

[23] M. Hofmann, *Extensional Constructs in Intensional Type Theory*, Springer-Verlag, London, 1997.